On some aspects of sensitivity analysis in AHP –an Illustration

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Abstract

This paper aims at giving an application of Analytical Hierarchy Process (AHP, a Multi Criteria Decision Making method) . Here AHP is applied for selection of a student from an Engineering college who is eligible for All Round Excellence Award for the year 2004-05 by taking subjective judgments of decision maker into consideration. Sensitivity analysis was applied to the problem and it was observed Alternative A_2 is the most sensitive to changes.

Key Words : Analytic Hierarchy Process, Deterministic Decision Making, Multi-Criteria Decision Making, Sensitivity Analysis, and Weighted Sum Model.

1 INTRODUCTION:

1.1 The Analytic Hierarchy Process (AHP):

The Analytic Hierarchy Process (AHP) is a multi-criteria decision making approach and was introduced by Saaty .The AHP has attracted the interest of many researchers mainly due to the nice mathematical properties of the method and the fact that the required input data are rather easy to obtain. The AHP is a decision support tool which can be used to solve complex decision problems. It uses a multi-level hierarchical structure of objectives, criteria, sub criteria, and alternatives. The data are derived by using a set of pair wise comparisons. These comparisons are used to obtain the weights of importance of the decision criteria, and the relative performance measures of the alternatives in terms of each individual decision criterion. If the comparisons are not perfectly consistent, then it provides a mechanism for improving consistency.

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A. Establishment of a structural Hierarchy

A complex decision is to be structured in to a hierarchy descending from an overall objective to various criteria, sub criteria till the lowest level. The overall goal of the decision is represented at the top level of the hierarchy. The criteria and the sub criteria, which contribute to the decision, are represented at the intermediate levels. Finally the decision alternatives are laid down at the last level of the hierarchy. According to Saaty (2000), a hierarchy can be constructed by creative thinking, recollection and using people's perspectives.

B. Establishment of comparative judgments

Once the hierarchy has been structured, the next step is to determine the priorities of elements at each level. A set of comparison matrices of all elements in a level with to respect to an element of

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the immediately higher level are constructed. The pair wise comparisons are given in terms of how much element A is more important than element B. The preferences are quantified using a nine – point scale.

C. Synthesis of priorities and measurement of consistency

The pair wise comparisons generate the matrix of rankings for each level of the hierarchy after all matrices are developed and all pair wise comparisons are obtained, Eigen vectors (relative weights) are obtained.

Eigen Vector Method: Suppose we wish to compare a set of 'n' objects in pairs according to their relative weights. Denote the objects by A1,A2,…..An and their weights by w1,w2,…..wn.

Our problem takes the form Aw= nw. We started with the assumption that w was given. But if we only had A and wanted to recover w, we would have to solve the system (A- nI) w = 0 in the unknown w. This has a nonzero solution if n is an eigenvalue of A, i.e., it is a root of the characteristic equation of A. But A has unit rank since every row is a constant multiple of the first row. Thus all the eigenvalue λ i ,i=1,2,…..n of A are zero except one. Also it is known that $\sum_{i=1}^{n} \lambda_i$ = tr(A) = n, and λi = 0, λi ≠ λmax . The solution w of this problem is any column of A. These solutions differ by a multiplicative constant. However, this solution is normalized so that its components sum to unity. The result is unique solution no matter which column is used. The matrix A satisfies the cardinal consistency property The consistency ratio is calculated as per the following steps

i) Calculate the Eigen vector or the relative weights and λmax for each matrix of order n

ii)Compute the consistency index for each matrix of order n by the formulae CI= (λmax – n) / (n – 1)

iii) The consistency ratio is then calculated using the formulae CR = CI / RI ,where RI is a known random consistency index obtained from a large number of simulation runs and varies depending upon the order of the matrix .

1.2 On sensitivity analysis:

There is considerable research on sensitivity analysis for some operations research and management science models such as linear programming [1]. The analysis is done after the optimal decision is determined. However, research on sensitivity analysis in deterministic multi-criteria decision making (MCDM) models is limited. As a related comment it should also be stated here that Expert Choice (1990), a software package on the AHP, performs a type of elementary sensitivity analysis. The user has the option to graphically alter the weights of the decision criteria and see on the screen how the rankings of the alternatives will change. However, the issue of criteria sensitivity analysis is not studied systematically. Moreover, Expert Choice does not offer any means for studying the effects of changes on the measures of performance of the alternatives.

In decision making the weights assigned to the decision criteria attempt to represent the genuine importance of the criteria. When criteria cannot be expressed in quantitative terms (such as cost, weight, volume, etc.), then it is difficult to represent accurately the importance of these criteria. In a situation like this, the decision making process can be improved considerably by identifying the critical criteria and then re-evaluate more accurately the weights of these criteria.

The decision maker can make better decisions if he/she can determine how critical each criterion is. In other words, how sensitive the actual ranking of the alternatives is to changes on the current weights of the decision criteria. In the first problem we determine how critical each criterion is, by performing a sensitivity analysis on the weights of the criteria. This sensitivity analysis approach determines, what is the smallest change in the current weights of the criteria, which can alter the existing ranking of the alternatives. Next, we use a similar concept to determine how critical the various performance measures of the alternatives(in terms of a single decision criterion at a time) are in the ranking of the alternatives.

Some decision methods (for instance, the AHP) require that the aij values represent relative importance. Given the above data and a decision making method, the objective of the decision maker is to find the best (i.e., the most preferred) alternative or to rank the entire set of alternatives. Let Pi(for i= 1,2,3,...,M)represent the final preference of alternative Ai when all decision criteria are considered. Different decision methods apply different procedures in calculating the values Pi.Without loss of generality, it can be assumed (by a simple rearrangement of the indexes) that the M alternatives are arranged in such a way that the following relation (ranking) is satisfied (that is, the first alternative is always the best alternative and so on):

P1 ≥ P2≥ P3≥..≥ PM.

2 METHODOLOGY

In this section we discuss about a method , WSM and some important definitions that govern the Sensitivity analysis.

2.1 The Weighted Sum Method:

The simplest and still the widest used MCDM method is the weighted sum model (WSM). The preference Pi of alternative Ai (i= 1,2,3,...,M)is calculated according to the following formula (Fishburn, 1967): P

$P_{ij} = \sum_{j=1}^{N} a_{ij}\ w_j$, for i ' 1,2,3,...,M .

Therefore, in the maximization case, the best alternative is the one which corresponds to the largest preference value. The supposition which governs this model is the additive utility assumption. However, the WSM should be used only when the decision criteria can be expressed in identical units of measure (e.g., only dollars, or only pounds, or only seconds, etc.).

2.2 Some Important Definitions:

DEFINITION 1: Let $\delta_{k,i,j}$ ($1 \le i < j \le M$ and $1 \le k \le N$) denote the minimum change in the

current weight W_k of criterion *Ck* such that the ranking of alternatives *Ai* and *Aj* will be reversed.

Also, define as:

$\delta^{I}_{k,i,j} = \delta_{k,i,j} \times 100/Wk$, for any $1 \le i < j \le M$ and $1 \le k \le N$. That is, $\delta^{I}_{k,i,j}$ expresses changes in relative terms. It is possible for a given pair of alternatives and a decision criterion, the critical change to be infeasible.The most critical criterion is defined in two possible ways (recall that from relations (2) alternative *A1* is always assumed to be the best alternative). The first of these two definitions (i.e., definition 2) applies when one is interested only in changes in the best alternative, while the second definition (i.e., definition 3) applies when one is interested in changes in the ranking of any alternative. Recall that *s*stands for the absolute value function (e.g., *-5* = +5).

DEFINITION 2: The Percent-Top (or PT) critical criterion is the criterion which corresponds to the smallest $|\delta^{I} k,i,j|$ ($1 \le j \le M$ and $1 \le k \le N$) value.

DEFINITION 3: The Percent-Any (or PA) critical criterion is the criterion which corresponds to the smallest $|\delta^{I} k,i,j|$, ($1 \le i < j \le M$ and $1 \le k \le N$) value.

It can be recalled that in this paper we adopt the definitions which correspond to relative changes. The following two definitions express how critical a given decision criterion is.

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DEFINITION 4: The **criticality degree of criterion *Ck***, denoted as *D* /

k, is the smallest percent amount by which the current value of *Wk* must change, such that the existing ranking of the alternatives will change. That is, the following relation is true:

$$D_k' = \min_{1 \le i < j \le M} \{|\delta^{I} k, i, j|\} \quad , \text{for all } N \ge k \ge 1.$$

DEFINITION 5: The **sensitivity coefficient of criterion *Ck***, denoted as ***sens*(*Ck*)**, is the reciprocal of its criticality degree. That is, the following relation is true:

$$sens(Ck)' = \frac{1}{Dk'} \quad , \text{for any } N \ge k \ge 1.$$

If the criticality degree is infeasible (i.e., impossible to change any alternative rank with any

weight change), then the sensitivity coefficient is set equal to zero.

3 ILLUSTRATION

Here AHP is applied for selection of a student from an Engineering college who is eligible for *All Round Excellence Award* for the year 2004-05 by taking subjective judgments of decision maker into consideration. Seven criteria were identified for getting this award and the alternatives are the five Branches of an Engineering college, in the state of Andhra Pradesh, INDIA.

Table 1 a) CRITICAL DEGREE OF ALTERNATIVES

	c1	**c2**	**c3**	**c4**	**c5**	**c6**	**c7**
A1	38.16(A5)	NF	66.8(A5)	51.54(A5)	70.85(A5)	73.37(A5)	30.39(A5)
A2	2.11(A4)	17.5	2.96(A4)	3.7(A4)	2.9(A4)	3.09(A4)	3.78(A4)
A3	NF	NF	NF	NF	NF	NF	NF
A4	17.2(A1)	NF	21.8(A1)	18.5(A1)	22.6(A1)	20.35(A1)	7.70(A1)
A5	NF	NF	NF	NF	NF	NF	52.7(A3)

Table 1 b) CRITICAL DEGREE OF ALTERNATIVES

c8	**c9**	**c10**	**c11**	**c12**	**c13**	**c14**	**c15**
28.91(A5)	19.53(A5)	26.78(A5)	34.46(A5)	39.1(A5)	34.7(A5)	24.2(A5)	22.2(A5)
4.3(A4)	3.06(A4)	1.9(A4)	2.2(A4)	3.01(A4)	2.2(A4)	**1.4(A4)**	2.01(A4)
NF	NF	NF	NF	-34.8(A2)	NF	NF	NF
8.08(A1)	6.5(A1)	6.91(A1)	9.7(A1)	13.2(A1)	9.83(A1)	7.68(A1)	7.25(A1)
77.8(A3)	70.9(A3)	68.8(A3)	-85.8(A1)	-46.3(A1)	-86.0(A1)	96.8(A3)	-49.3(A1)

Table 2 a) SENSITIVITY COEFFICIENT OF ALTERNATIVES

	c1	**c2**	**c3**	**c4**	**c5**	**c6**	**c7**
A1	0.02	NF	0.014	0.019	0.014	0.013	0.032
A2	0.47	0.05	0.337	0.27	0.344	0.323	0.264
A3	NF	NF	NF	NF	NF	NF	NF
A4	0.05	NF	0.045	0.054	0.04	0.04	0.129
A5	NF	NF	NF	NF	NF	NF	0.018

Table 2 b) SENSITIVITY COEFFICIENT OF ALTERNATIVES

c8	**c9**	**c10**	**c11**	**c12**	**c13**	**c14**	**c15**
0.034	0.051	0.037	0.029	0.025	0.028	0.041	0.045
0.232	0.326	0.526	0.454	0.332	0.454	0.714	0.497
NF	NF	NF	NF	-0.028	NF	NF	NF
0.123	0.153	0.144	0.103	0.075	0.101	0.13	0.137
0.012	0.014	0.014	-0.011	-0.021	-0.011	0.01	-0.02

4 CONCLUSIONS

The study says that the alternative with least critical degree is the most sensitive alternative. From the above illustration , it is clear that A2 is most sensitive .

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